4.3 USING ITERATION TO MODEL POPULATION GROWTH (Optional Section)

Although I shall henceforth adopt the habit of referring to the variable X as "the population," there are countless situations outside population biology where . . . [iteration of functions] *applies. . . . Examples in economics include models for the relationship between commodity quantity and price, for the theory of business cycles, and for the temporal sequences generated by various other economic quantities.*

. . . I would therefore urge that people be introduced to, say, [the iteration process for $f(x) = kx(1 - x)$] *early in their mathematical education. This equation can be studied phenomenologically by iterating it on a calculator, or even by hand. Its study does not involve as much conceptual sophistication as does elementary calculus. Such study would greatly enrich the student's intuition about nonlinear systems.*

—Biologist Robert M. May, "Simple mathematical models with very complicated dynamics," *Nature,* vol. 261 (1976), pp. 459–467.

The size of a population or its genetic makeup may change from one generation to the next. In the study of *discrete dynamics* we use functions and the iteration process (from Section 3.5) to investigate and analyze changes such as these that occur over discrete intervals of time. We begin by introducing the notion of a *fixed point* of a function.

If we start with a function f and an input x, it's usually not the case that $f(x)$ turns out to be the same as x itself. That is, usually, the output is not the same as the input. But sometimes this does happen. Take, for example, the function $f(x) = 3x - 2$ and the input $x = 1$. Then we have

$$f(1) = 3(1) - 2 = 1$$

So for this particular function the input $x = 1$ is an instance where "input = output." The input $x = 1$ in this case is called a *fixed point* of the function $f(x) = 3x - 2$. In the box that follows, we give the general definition of a fixed point.

Definition **Fixed Point of a Function**

A fixed point of a function f is an input x in the domain of f such that

$$f(x) = x$$

EXAMPLE

Both 0 and 1 are fixed points for $f(x) = x^2$ because $f(0) = 0^2 = 0$ and $f(1) = 1^2 = 1$.

EXAMPLE 1 **Finding Fixed Points**

Find the fixed points (if any) for each function:

(a) $f(x) = 1 - x$; **(b)** $g(x) = 1 + x$; **(c)** $h(x) = x^2 - x - 3$.

SOLUTION **(a)** We're looking for a number x such that $f(x) = x$. In view of the definition of f, this last equation becomes

$$1 - x = x$$

and therefore

$$x = \frac{1}{2}$$

This result shows that the function f has one fixed point; it is $x = 1/2$.

(b) If x is a fixed point of $g(x) = 1 + x$, we have

$$1 + x = x$$

But then subtracting x from both sides of this last equation yields $1 = 0$, which is impossible. We conclude from this that there is no fixed point for the function $g(x) = 1 + x$.

(c) The fixed points (if any) are the solutions of the quadratic equation $x^2 - x - 3 = x$. Subtracting x from both sides of this equation, we have

$$\begin{aligned} x^2 - 2x - 3 &= 0 \\ (x - 3)(x + 1) &= 0 \end{aligned}$$

Looking at this last equation, we can see that there are two roots: 3 and -1. Each of these numbers is a fixed point for the given function $h(x) = x^2 - x - 3$. That is, $h(3) = 3$ and $h(-1) = -1$. [You should verify each of these last two statements for yourself by actually computing $h(3)$ and $h(-1)$.] ■

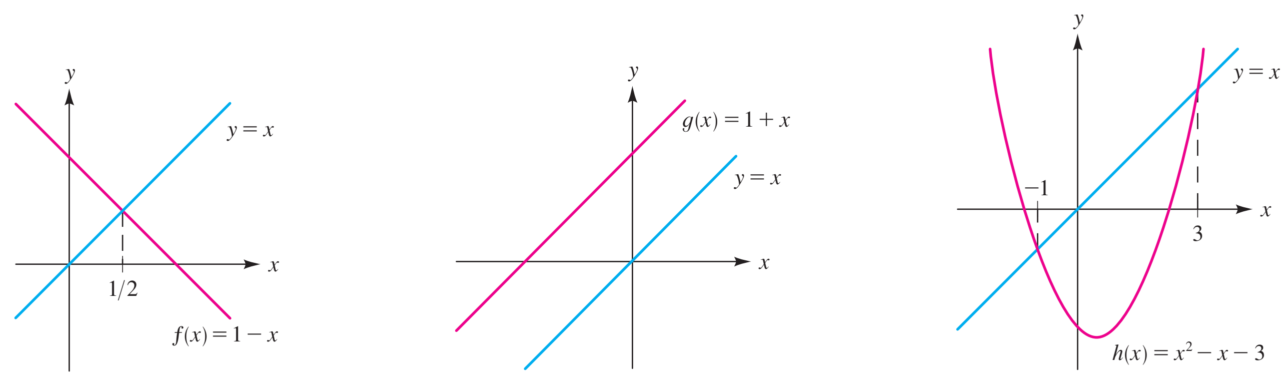
A fixed point of a function can be interpreted geometrically: It is the x-coordinate of a point where the graph of the given function intersects the line $y = x$. Figure 1 shows the fixed points for the functions in the example that we've just completed.

Fixed points are related to the iteration process in several ways. Suppose that a number a is a fixed point of the function f. Then by definition we have $f(a) = a$, which says that the first iterate of a is equal to a itself. Similarly, all of the subsequent iterates of the fixed point a will be equal to a. For instance, for the second iterate we have

$$\begin{aligned} f(f(a)) &= f(a) && \text{substituting } a \text{ for } f(a) \text{ on the left-hand side} \\ &= a && \text{again because } f(a) = a \end{aligned}$$

This shows that the second iterate is equal to a. The same type of calculation will show that any subsequent iterate of the fixed point a is equal to a.

Another connection between fixed points and iteration is this: for some functions, a fixed point can be a "target value" for other iterates. We'll explain this using Figure 2.



(a) The fixed point of f is 1/2.

(b) The function g has no fixed points.

(c) The fixed points of h are -1 and 3.

Figure 1
Fixed points for the functions in Example 1.

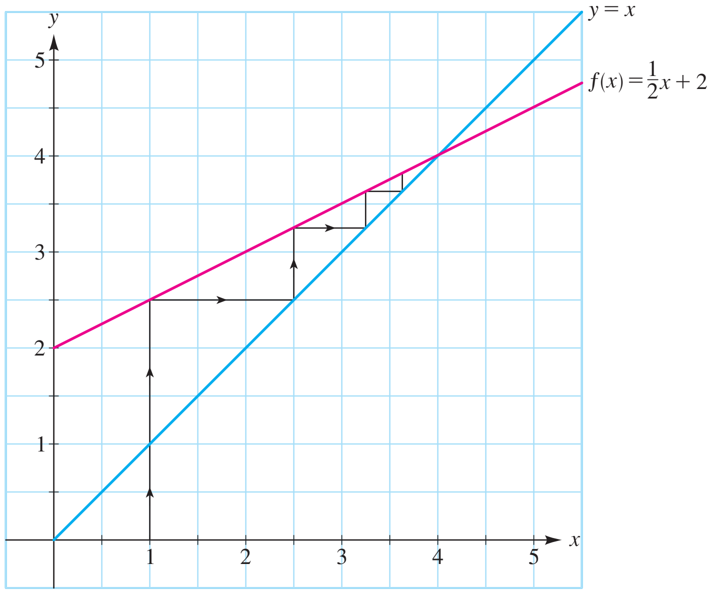


Figure 2
The first four steps in the iteration process for $f(x) = \frac{1}{2}x + 2$ with $x_0 = 1$.

Figure 2 shows the first four steps in the iteration process for $f(x) = \frac{1}{2}x + 2$ with $x_0 = 1$. (See Section 3.5 if you need to review graphical iteration.) As indicated in Figure 2, the input 4 is a fixed point for the function f, and the iteration process follows a staircase pattern that approaches the point (4, 4). We say in this case that the iterates of $x_0 = 1$ **approach** the fixed point 4 and that this target value 4 is an **attracting fixed point** of the function f. Table 1 gives you a more numerical look at what is meant by saying that the iterates approach the target value 4.

In the table, notice, for example, that

x_5 differs from 4 by less than 0.1

x_{10} differs from 4 by less than 0.01

and x_{15} differs from 4 by less than 0.0001

What's important here is that the differences between the iterates and 4 can be made as small as we please by carrying out the iteration process sufficiently far. The idea

TABLE 1 The Iterates Approach 4

x_1	2.5		
x_2	3.25		
x_3	3.625		
x_4	3.8125	x_{11}	3.9985 . . .
x_5	3.906 . . .	x_{12}	3.99926 . . .
x_6	3.953 . . .	x_{13}	3.99963 . . .
x_7	3.976 . . .	x_{14}	3.99981 . . .
x_8	3.988 . . .	x_{15}	3.999908 . . .
x_9	3.9941 . . .	x_{20}	3.9999971 . . .
x_{10}	3.9970 . . .	x_{25}	3.999999910 . . .

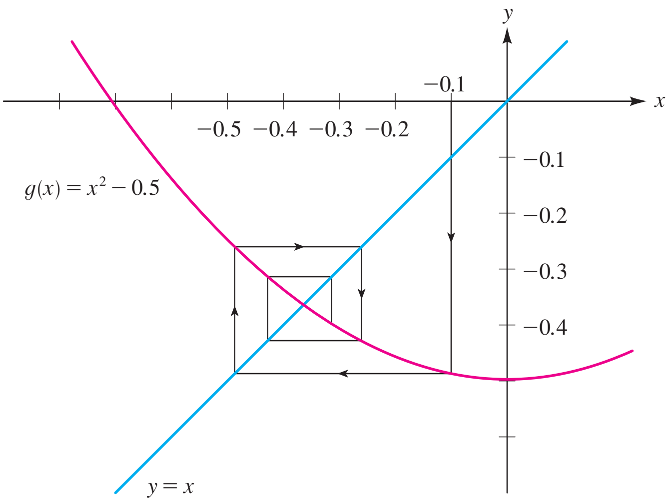


Figure 3
The iterates of $x_0 = -0.1$ under the function $g(x) = x^2 - 0.5$ approach the attracting fixed point $\frac{1}{2}(1 - \sqrt{3}) \approx -0.366$.

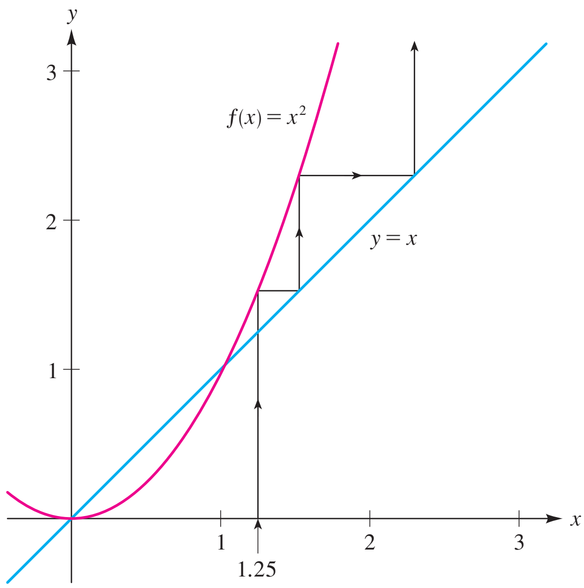


Figure 4
The iterates of $x_0 = 1.25$ under the function $f(x) = x^2$ move away from the repelling fixed point 1.

of a target value or *limit* is made more precise in calculus. But for our purposes, Figure 2 and Table 1 will certainly give you an intuitive understanding of the idea and what we mean by saying that the iterates approach 4.

Figures 3 and 4 show two more ways in which the iteration process may relate to fixed points. In Figure 3 there is an attracting fixed point for the iterates of $x_0 = -0.1$, but this time the iteration process approaches the fixed point through a spiral pattern rather than a staircase pattern. To find the fixed point (and thereby determine the number that the iterates are approaching), we need to solve the quadratic equation $x^2 - 0.5 = x$. As you should check for yourself by means of the quadratic formula and then a calculator, the relevant root here is $(1 - \sqrt{3})/2 \approx -0.366$. Notice that this value is consistent with Figure 3. In Figure 4 the fixed point 1 is a **repelling fixed point** for the iterates of $x_0 = 1.25$. As Figure 4 indicates, the iterates of $x_0 = 1.25$ move farther and farther away from the value 1. Indeed, as you can check with a calculator, the first five iterates of 1.25 are as follows (we're rounding to two decimal places):

$$x_1 \approx 1.56 \qquad x_2 \approx 2.44 \qquad x_3 \approx 5.96 \qquad x_4 \approx 35.53 \qquad x_5 \approx 1262.18$$

The iteration process for functions is often applied in the study of population growth. The word "population" here is used in a general sense. It needn't refer only to human populations. For instance, biological or ecological studies may involve animal, insect, or bacterial populations. (Also, see the quotation at the beginning of this section.) The following equation defines one type of quadratic function that has been studied extensively in this context:

$$f(x) = kx(1 - x) \tag{1}$$

In using this idealized model, we assume that the population size is measured by a number between 0 and 1, where 1 corresponds to the maximum possible population size in the given environment and 0 corresponds to the case in which the population has become extinct. We start with a given input x_0 $(0 \leq x_0 \leq 1)$ that represents the fraction of the maximum population size that is initially present. For instance, if the maximum possible population of catfish in a pond is 100 and initially there were 70 catfish, then we would have $x_0 = 70/100 = 0.7$.

The next basic assumption in using equation (1) to model population size is that the iterates of x_0 represent the fraction of the maximum possible population present after each successive time interval. That is,

$f(x_0) = x_1$ is the fraction of the maximum population after the first time interval

$f(x_1) = x_2$ is the fraction of the maximum population after the second time interval

and, in general,

$f(x_{n-1}) = x_n$ is the fraction of the maximum population after the nth time interval

It is important to note that the function $f(x)$ does *not* represent the size of the population. The population size after n time intervals is given by

$$x_n \cdot (\text{the maximum population}) = f(x_{n-1}) \cdot (\text{maximum population})$$

In a given study, the time intervals might be measured, for example, in years, in months, or in breeding seasons. The constant k in equation (1) is the *growth parameter*; it is related to the rate of growth of the particular population being studied. Science writer James Gleick has described k this way: "In a pond, it might correspond to the fecundity of the fish, the propensity of the population not just to boom but also to bust. . . ." [*Chaos: Making a New Science* (New York: Viking Penguin, Inc., 1988)]

EXAMPLE 2 Using Iteration in Analyzing Population Size

In the Mississippi Delta region, many farmers have replaced unproductive cotton fields with catfish ponds. Suppose that a farmer has a catfish pond with a maximum population size of 500 and that initially the pond is stocked with 50 catfish. Also, assume that the growth parameter for this population is $k = 2.9$, so that equation (1) becomes

$$f(x) = 2.9x(1 - x) \tag{2}$$

Finally, assume that the time intervals here are breeding seasons.

(a) What is the value for x_0?

(b) Use Figure 5 to estimate the iterates x_1 through x_5. Then use a calculator to compute these values. Round the final answers to three decimal places. Interpret the results.

(c) As indicated in Figure 5, the iteration process is spiraling in on a fixed point of the function. (Figure 6 later in this section will demonstrate this in greater detail.) Find this fixed point and interpret the result.

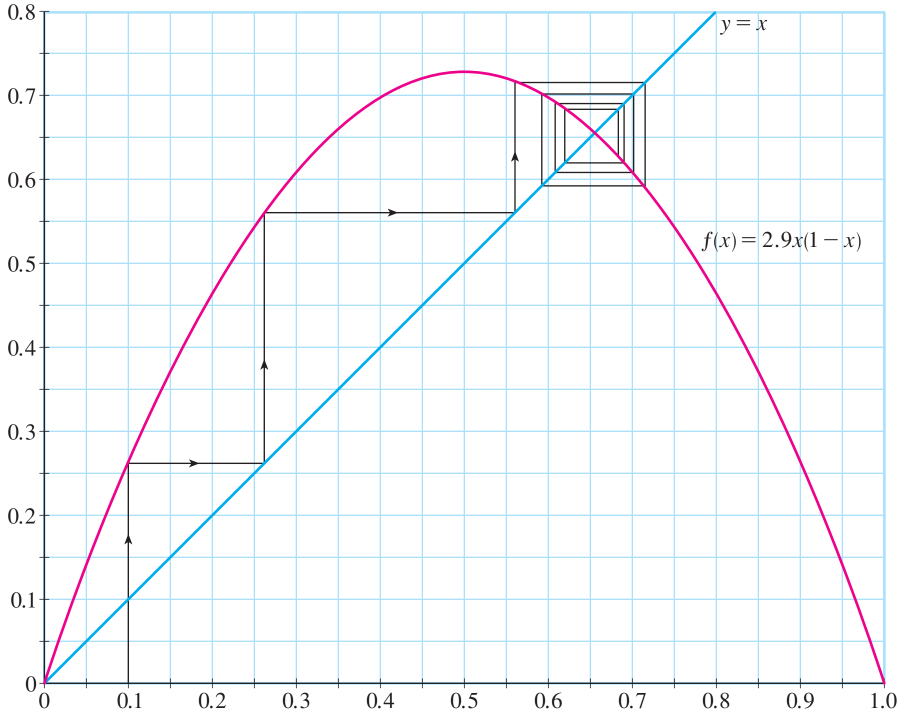


Figure 5
The first ten iterations of $x_0 = 0.1$ under $f(x) = 2.9x(1 - x)$.

SOLUTION **(a)** $x_0 = \dfrac{\text{initial population}}{\text{maximum population}} = \dfrac{50}{500} = 0.1$

(b) In looking at Figure 5, it appears that x_1, the first iterate of x_0, is between 0.25 and 0.30, much closer to the former number than the latter. As an estimate, let's say that x_1 is about 0.26. This and the other estimates are given in Table 2. ***Suggestion:*** Make the estimates for yourself before looking at the estimates we give. Some slight discrepancies are okay. In the bottom row of Table 2 are the values of the iterates obtained using a calculator. You should verify these for yourself. (There should be no discrepancies here.)

The results in Table 2 tell us what is happening to the population through the first five breeding seasons. From an initial population of 50 catfish, the population size steadily increases through the first three breeding seasons (the numbers in the table are getting bigger). The population size then drops after the fourth breeding season and goes back up after the fifth season. (These facts can be deduced from Figure 5, as well as from Table 2.) To compute the actual numbers of

TABLE 2 **Iterates of $x_0 = 0.1$ Under the Function $f(x) = 2.9x(1 - x)$ (calculator values rounded to three decimal places)**

	x_1	x_2	x_3	x_4	x_5
From Graph	0.26	0.56	0.71	0.58	0.70
From Calculator	0.261	0.559	0.715	0.591	0.701

TABLE 3 Catfish Population

n	0	1	2	3	4	5
Number of Fish After *n* Breeding Seasons	50	131	280	358	296	351

fish, we need to multiply each number in the bottom row of Table 2 by 500. (Remember that the iterates in Table 2 represent fractions of the maximum possible population size 500.) For example, to compute the actual number of fish at the end of the third breeding season, we multiply the initial population size of 500 by x_3:

$$\begin{aligned} x_3 \times 500 &\approx 0.715 \times 500 \\ &\approx 358 \text{ catfish} \end{aligned}$$

The number of catfish at the end of each of the other breeding seasons is obtained in the same manner. See Table 3; use your calculator to check each of the entries in the table.

(c) The fixed point we are looking for occurs when the parabola in Figure 5 intersects the line $y = x$. So, following the method in Example 1, we need to solve the equation $2.9x(1 - x) = x$. We have

$$\begin{aligned} 2.9x(1 - x) &= x \\ -2.9x^2 + 1.9x &= 0 \\ x(-2.9x + 1.9) &= 0 \end{aligned}$$

Therefore, $x = 0$ or $-2.9x + 1.9 = 0$, that is,

$$x = \frac{-1.9}{-2.9} = \frac{19}{29} \approx 0.655 \quad \text{using a calculator and rounding to three decimal places}$$

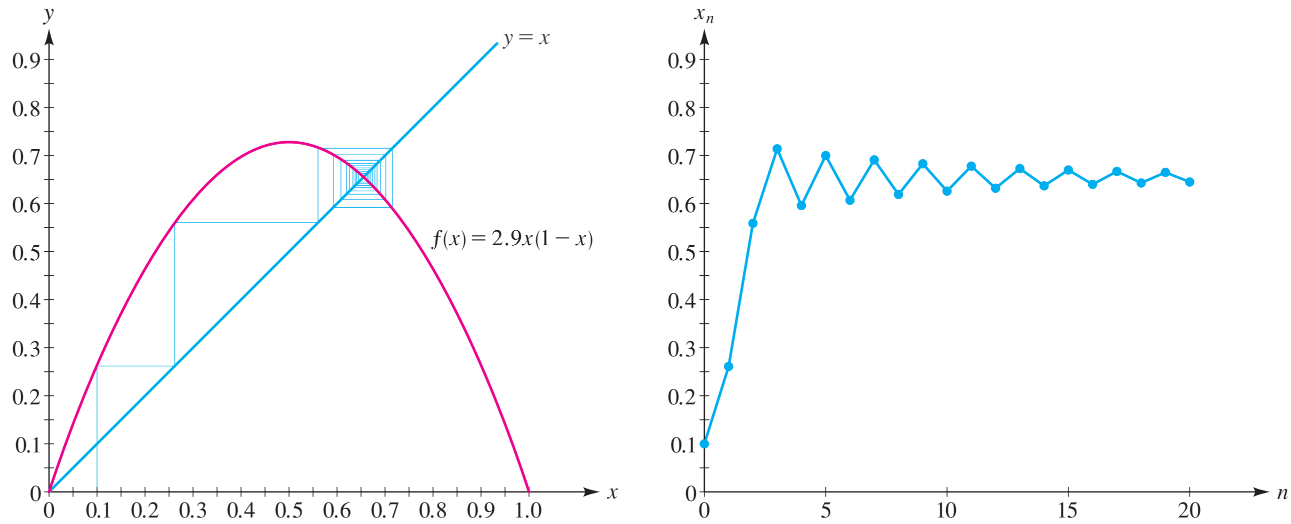
This shows that there are two fixed points for the function f, namely, 0 and 19/29. Looking at Figure 5, we know that $19/29$ is the fixed point that we are interested in here, not 0. So the iterates of $x_0 = 0.1$ are approaching the value $19/29$, which is approximately 0.655.

We now summarize and interpret the results. There were initially 50 catfish in a pond that could hold at most 500. As we saw in part (b), the population size increases over the first three breeding seasons. After this, as Figure 5 shows, the population oscillates up and down but draws closer and closer to an equilibrium population corresponding to the fixed point $x = 19/29$. This equilibrium population is

$$\frac{19}{29} \times 500 \approx 328 \text{ catfish}$$

■

We conclude this section with some pictures indicating only three of the many possibilities that can arise in the iteration of quadratic functions of the form $f(x) = kx(1 - x)$. Figure 6 concerns the function with growth parameter $k = 2.9$ that we used in the catfish example: $f(x) = 2.9x(1 - x)$. In Figure 6(a) we've carried



(a) The iteration process for the first 20 iterates.

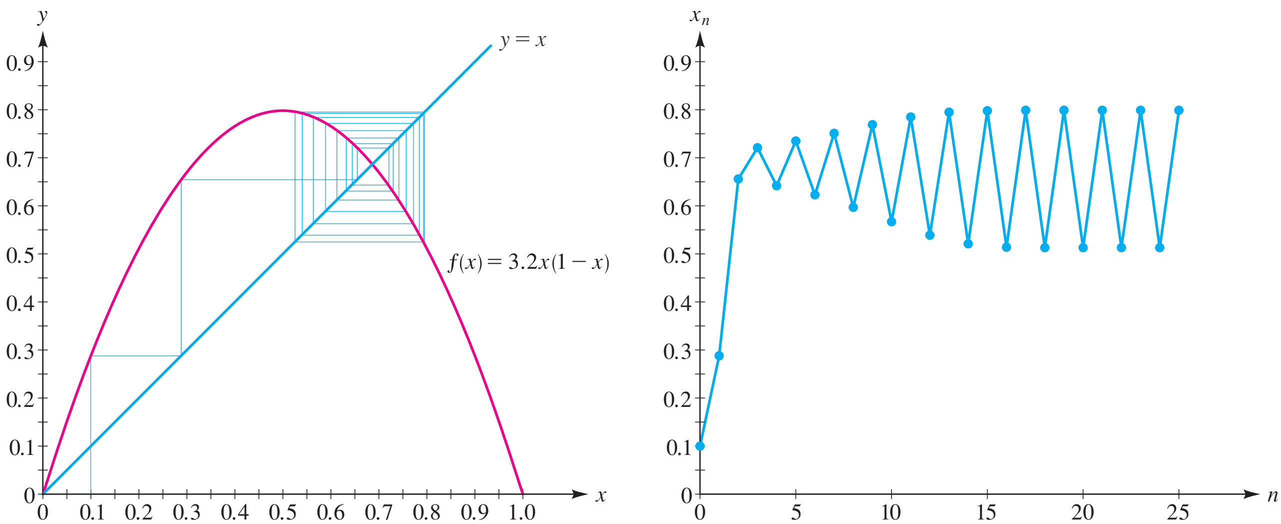
(b) The iterates approach a number between 0.6 and 0.7.

Figure 6
The iteration of $f(x) = kx(1 - x)$ with growth parameter $k = 2.9$ and $x_0 = 0.1$.

out the iteration of $x_0 = 0.1$ through the twentieth iterate. (In the catfish example, Figure 5 goes only as far as the tenth iterate.) Figure 6(a) indicates quite clearly that the iterates are indeed approaching a fixed point of the function. Figure 6(b) presents another way to visualize the long-term behavior of the iterates. Values of n are marked on the horizontal axis, values of the iterates x_n are marked on the vertical axis, and the points with coordinates (n, x_n) are then plotted. For example, since $x_0 = 0.1$, we plot the point $(0, 0.1)$; and since $x_1 = 0.261$, we plot the point $(1, 0.261)$. The line segments in Figure 6(b) are drawn in only to help the eye see the pattern that is emerging. Three facts that can be inferred from Figure 6(b) are as follows: After the first few iterates, the iterates oscillate up and down; the magnitude of the oscillations is decreasing; and, in the long run, the iterates are approaching a number between 0.6 and 0.7. (In Example 2 we found this value to be approximately 0.655.)

Unlike the iteration pictured in Figure 6(a), Figure 7(a) shows a case in which the iteration process is spiraling away from, rather than toward, a fixed point. Again, we've used the initial input $x_0 = 0.1$, but this time the growth parameter is $k = 3.2$. As indicated in Figure 7(b), the long-term behavior of the iterates becomes quite predictable: they alternate between two values. Figure 7(b) shows that the smaller of these two values is between 0.5 and 0.6, while the larger is approximately 0.8. Exercise 36 gives you formulas for computing these two limiting values for the iterates. (They turn out to be, approximately, 0.513 and 0.799.)

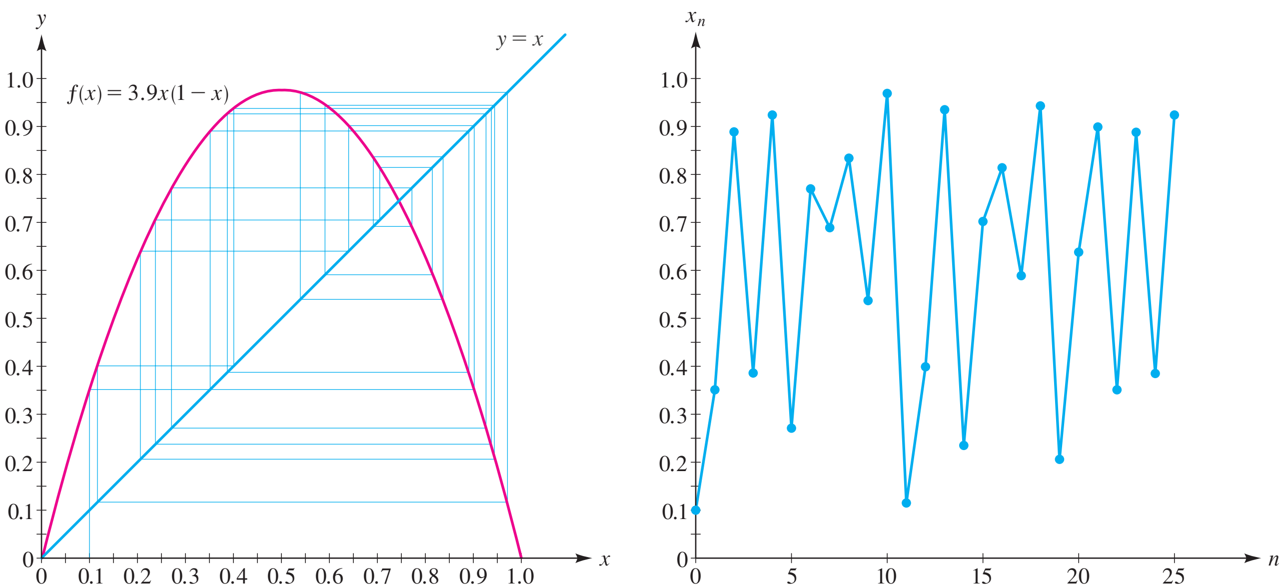
In Figure 8, once again we take the initial input to be $x_0 = 0.1$, but this time the growth parameter is $k = 3.9$. Now the iterates seem to fluctuate widely with no apparent pattern, in sharp contrast to the previous two figures, in which there were clear patterns. Phenomena such as this are the subject of **chaos theory,** a new branch of twentieth-century mathematics with wide application. [For a readable and nontechnical introduction to this relatively new subject, see James Gleick's *Chaos: Making a New Science* (New York: Viking Penguin, Inc., 1988). For a little more detail on the mathematics, see the paperback by Donald M. Davis, *The Nature and Power of Mathematics* (Princeton, N.J.: Princeton University Press, 1993), pp. 314–363.]



(a) The iteration process for the first 25 iterates.

(b) The iterates alternately approach two numbers. These numbers are approximately 0.513 and 0.799.

Figure 7
The iteration of $f(x) = kx(1 - x)$ with growth parameter $k = 3.2$ and $x_0 = 0.1$.



(a) The iteration process for the first 25 iterates.

(b) The iterates fluctuate widely; no clear pattern emerges.

Figure 8
The iteration of $kx(1 - x)$ with growth parameter $k = 3.9$ and $x_0 = 0.1$.

EXERCISE SET 4.3

A

In Exercises 1–16, find all real numbers (if any) that are fixed points for the given functions.

1. $f(x) = -4x + 5$
2. $g(x) = 3x - 14$
3. $G(x) = \frac{1}{2} + x$
4. $F(x) = (7 - 2x)/8$
5. $h(x) = x^2 - 3x - 5$
6. $H(t) = 3t^2 + 18t - 6$
7. $f(t) = t^2 - t + 1$
8. $F(t) = t^2 - t - 1$
9. $k(t) = t^2 - 12$
10. $K(t) = t^2 + 12$
11. $T(x) = 1.8x(1 - x)$
12. $T(y) = 3.4y(1 - y)$
13. $g(u) = 2u^2 + 3u - 4$
14. $G(u) = 3u^2 - 4u - 2$
15. $f(x) = 7 + \sqrt{x - 1}$
16. $f(x) = \sqrt{10 + 3x} - 4$

Ⓖ *In Exercises 17–22:*

(a) *Graph each function along with the line y = x. Use the graph to determine how many (if any) fixed points there are for the given function.*

(b) *For those cases in which there are fixed points, use the zoom-in capability of the graphing utility to estimate the fixed point. (In each case, continue the zoom-in process until you are sure about the first three decimal places.)*

17. $f(x) = x^3 + 3x + 2$
18. $g(x) = x^3 - 3x + 2$
19. $h(x) = x^3 - 3x - 3.07$
20. $k(x) = x^3 - 3x - 3.08$
21. $s(t) = t^4 + 3t - 2$
22. $u(t) = t^4 + 3t + 2$

23. This exercise refers to the function $g(x) = x^2 - 0.5$ in Figure 3 in the text.

(a) Use the quadratic formula to verify that one of the fixed points of this function is $(1 - \sqrt{3})/2$, then use your calculator to check that this is approximately -0.366.

(b) According to the text, the iterates of -0.1 approach the value determined in part (a). Use your calculator: which is the first iterate to have the digit 3 in the first decimal place?

(c) Use your calculator: Which is the first iterate to have the digit 6 in the second decimal place?

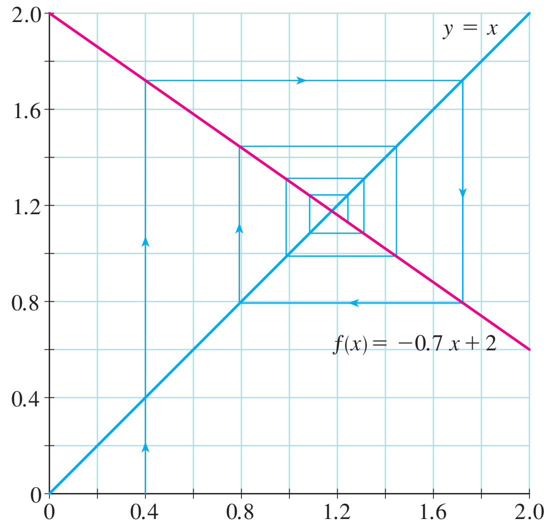
24. This exercise refers to the function $f(x) = x^2$ in Figure 4 in the text. According to the text, the iterates of 1.25 move farther and farther away from the fixed point 1. In this exercise you'll see that iterates of other points even closer to the fixed point 1 nevertheless are still "repelled" by 1.

(a) Let $x_0 = 1.1$. Use your calculator: Which is the first iterate to exceed 10? Which is the first iterate to exceed one million?

(b) Let $x_0 = 1.001$. Use your calculator: Which is the first iterate to exceed 2? Which is the first iterate to exceed one million?

(c) Let $x_0 = 0.99$. Compute x_1 through x_{10} to see that the iterates are indeed moving farther and farther away from the fixed point 1. What value are the iterates approaching? Is this value a fixed point of the function?

25. The accompanying figure shows the first eight steps in the iteration process for $f(x) = -0.7x + 2$, with $x_0 = 0.4$.



(a) Complete the following table. For the values obtained from the graph, estimate to the nearest one-tenth; for the calculator values, round the final answers to three decimal places.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
From Graph								
From Calculator								

(b) The figure shows that the iterates are approaching a fixed point of the function. Determine the exact value of this fixed point and then give a calculator approximation rounded to three decimal places.

(c) In the table for part (a), you used a calculator to compute the first eight iterates. Which of these iterates is the first to have the same digit in the first decimal place as the fixed point?

26. The figure on the next page shows the first six steps in the iteration process for $f(x) = 2.9x(1 - x)$, with $x_0 = 0.2$.

(a) Complete the following table. For the values obtained from the graph, estimate to the nearest 0.05, or closer if you can. For instance, to the nearest 0.05, the first iterate is 0.45. But, since the graph shows the iterate a bit above 0.45, the estimate 0.46 would be better. For the calculator work, round the final answers to three decimal places.

	x_1	x_2	x_3	x_4	x_5	x_6
From Graph						
From Calculator						

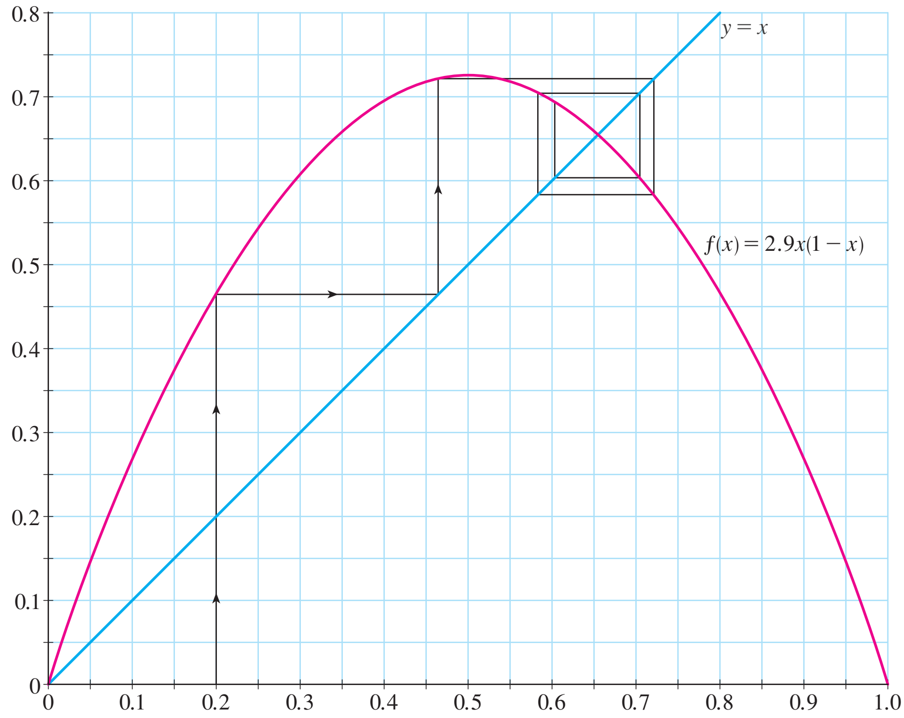


Figure for Exercise 26

(b) The figure shows that the iterates are approaching a fixed point of the function. Use the figure to estimate, to the nearest 0.05, a value for this fixed point. Then determine the exact value of this fixed point, and also give a calculator approximation rounded to three decimal places.

(c) In the table for part (a), you used a calculator to compute the first six iterates. Which of these iterates is the first to have the same digit in the first decimal place as the fixed point? ***Remark:*** Agreement in the second decimal place doesn't occur until the 26th iterate.

For Exercises 27 and 28, refer to the figure on page 4.3.12, which shows the first nine steps in the iteration process for $f(x) = 4x(1 - x)$, with $x_0 = 0.9$. (Qualitatively, note that this figure is quite different from those in the previous two exercises or in Figures 2 through 7 in the text. Here, no clear pattern in the iterates seems to emerge.)

27. Complete the following table. For the values obtained from the graph, estimate to the nearest 0.05, or closer if you can. For instance, to the nearest 0.05, the first iterate is 0.35. But since the graph shows the iterate a bit above 0.35, the estimate 0.36 would be better. For the calculator work, round the final answers to three decimal places.

	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9
From Graph									
From Calculator									

28. As in Exercise 27, we work with the function $f(x) = 4x(1 - x)$. Furthermore, in part (a) we use an input that is very close to the one in Exercise 27. As you will see, however, the results will be remarkably different. This phenomenon, whereby a very small change in the initial input results in a completely different pattern in the iterates, is called **sensitivity to initial conditions.** Sensitivity to initial conditions is one of the characteristic behaviors studied in chaos theory.

(a) For the initial input, use $x_0 = \frac{1}{8}(5 + \sqrt{5}) \approx 0.9045$. Note that this differs from the input in Exercise 27 by less than 0.01. (Exercise 36 shows how this seemingly off-the-wall input was obtained.) Use algebra (not a calculator!) to compute exact expressions for x_1 and x_2. What do you observe? What are x_3 and x_4? What's the general pattern here? ***Note:*** If you were to use a calculator rather than algebra for all of this, due to rounding errors you could miss seeing the patterns.

(b) Take $x_0 = 0.905$. Use a calculator to compute x_1 through x_{10}. Round the final results to three decimal places. Is the behavior of the iterates more like that in Exercise 27 or in part (a) of this exercise?

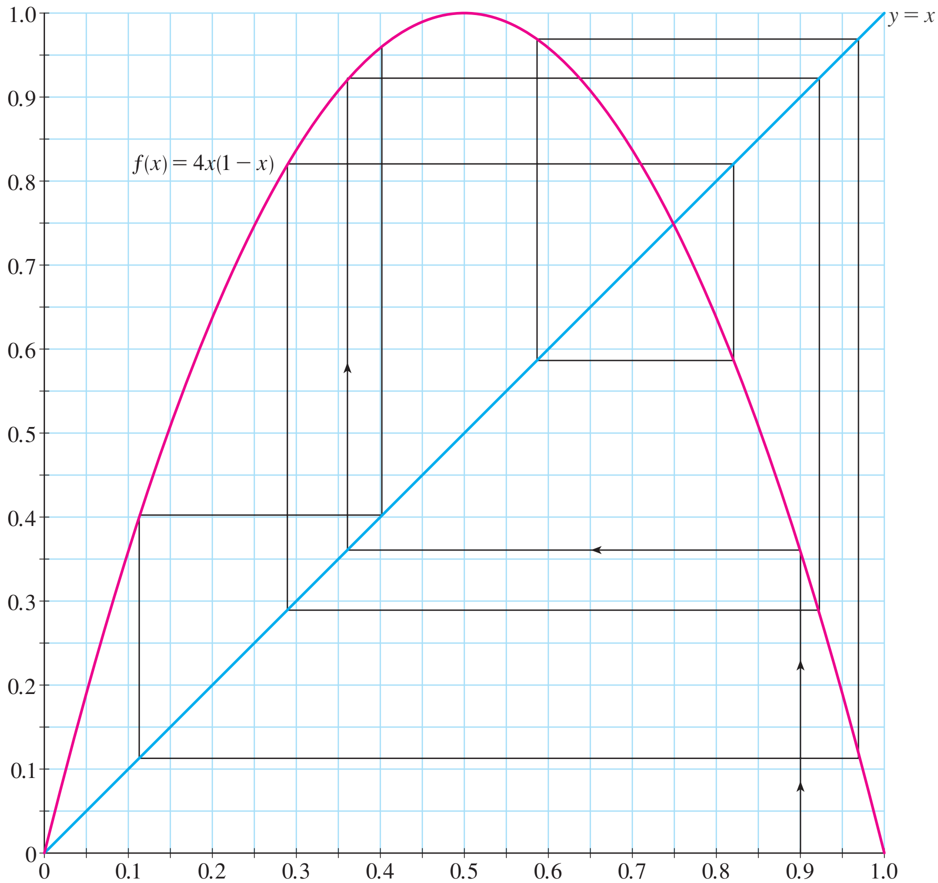


Figure for Exercises 27 and 28

As background for Exercises 29–32, you need to have read Example 2 in this section. As in Example 2, assume that there is a catfish pond with a maximum population size of 500 catfish. Also assume, unless stated otherwise, that the initial population size is 50 catfish (so that $x_0 = 0.1$).

29. (a) In Example 2 we used a growth parameter of $k = 2.9$ and we computed the first five iterates of $x_0 = 0.1$. Now (under these same assumptions) compute x_{21} through x_{25}, given that $x_{20} \approx 0.64594182$. Round your final answers to four decimal places.

(b) Use the results in part (a) to complete the following table. [Compare your table to Table 3 in the text; note that the population size continues to oscillate up and down, but now the sizes of the oscillations are much smaller. This provides additional evidence that the population size is approaching an equilibrium value. (In Example 2 we determined this equilibrium size to be about 328 catfish.)]

n	**20**	**21**	**22**	**23**	**24**	**25**
Number of Fish After *n* Breeding Seasons						

(c) In parts (a) and (b) and in Example 2 we worked with a growth parameter of 2.9, and we found that the iterates of $x_0 = 0.1$ were approaching a fixed point of the function. Now assume instead that the growth parameter is $k = 0.75$ (but, still, that $x_0 = 0.1$). Determine the iterates x_1 through x_5. (Round to five decimal places.) Are the iterates approaching a fixed point of the function $f(x) = 0.75x(1 - x)$? Interpret your results.

30. As in Example 2, take $x_0 = 0.1$, but now assume that the growth parameter is $k = 3$, so that equation (1) in the text becomes $f(x) = 3x(1 - x)$.

(a) Complete the following tables. (Round the final answers for the iterates to three decimal places.) Notice that after the third breeding season, the population oscillates up and down, as in Example 2.

n	**0**	**1**	**2**	**3**	**4**	**5**
x_n	0.1					
Number of Fish After *n* Breeding Seasons	50					

n	6	7	8	9	10
x_n					
Number of Fish After n Breeding Seasons					

(b) Complete the following table, given that $x_{100} \approx 0.643772529$. Round your final answers for the iterates to four decimal places. In your results, note that the population continues to oscillate up and down, but that the sizes of the oscillations are less than those observed in part (a).

n	101	102	103	104	105	106
x_n						
Number of Fish After n Breeding Seasons						

(c) The iterates that you computed in parts (a) and (b) are approaching a fixed point of the function $f(x) = 3x(1 - x)$. Find this fixed point and the corresponding equilibrium population size.

31. (a) As in Example 2, take $x_0 = 0.1$, but now assume that the growth parameter is $k = 3.1$, so that equation (1) in the text becomes $f(x) = 3.1x(1 - x)$. Complete the following three tables. For the third table, use the fact that $x_{20} \approx 0.56140323$. Round your final answers for the iterates to four decimal places.

n	0	1	2	3	4	5
x_n	0.1					
Number of Fish After n Breeding Seasons	50					

n	6	7	8	9	10
x_n					
Number of Fish After n Breeding Seasons					

n	21	22	23	24	25	26
x_n						
Number of Fish After n Breeding Seasons						

(b) Your results in part (a) will show that the population size is oscillating up and down, but that the iterates don't seem to be approaching a fixed point. Indeed, determine the (nonzero) fixed point of the function $f(x) = 3.1x(1 - x)$. Then note that the successive iterates in part (a) actually move farther and farther away from this fixed point.

(c) It can be shown that the long-term behavior of the iterates in this case resembles the pattern in Figure 7(b) on page 4.3.9; that is, the iterates alternately approach two values. Use the following formulas, with $k = 3.1$, to determine these two values a and b that the iterates are alternately approaching. (The formulas are developed in Exercise 36.) Round the answers to four decimal places. Check to see that your answers are consistent with the results in part (a).

$$a = \frac{1 + k + \sqrt{(k - 3)(k + 1)}}{2k}$$

$$b = \frac{1 + k - \sqrt{(k - 3)(k + 1)}}{2k}$$

(d) What are the two populations corresponding to these two numbers a and b?

32. As in Example 2, assume that the maximum population size of the pond is 500 catfish, but now suppose that the growth parameter is $k = 0.6$.

(a) Suppose that the initial population is again 50 catfish, so that $x_0 = 0.1$. Compute the first 10 iterates. Do they appear to be approaching a fixed point? Interpret the results.

(b) Follow part (a), but assume that the initial population is 450, so that $x_0 = 450/500 = 0.9$.

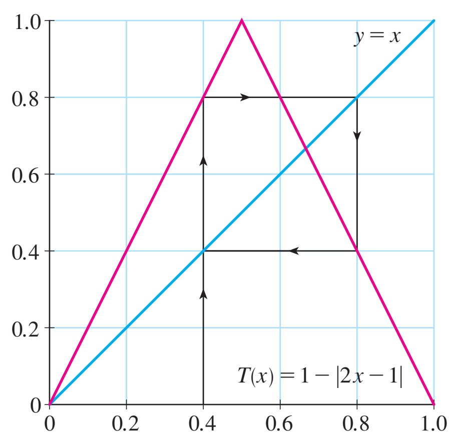
(c) What relationship do you see between the iterates in part (a) and in part (b)?

B

33. Suppose that c and d are inputs for a function g and that $g(c) = d$ and $g(d) = c$. Then we say the set $\{c, d\}$ is a **2-cycle** for the function g.

(a) Assuming that $\{c, d\}$ is a 2-cycle for the function g, list the first six iterates of c and the first six iterates of d. Describe in a complete sentence or two the pattern you see.

(b) The following figure shows the iteration process for the function $T(x) = 1 - |2x - 1|$, with initial input $x_0 = 0.4$. Use the figure to list the first six iterates of 0.4 under the function T.

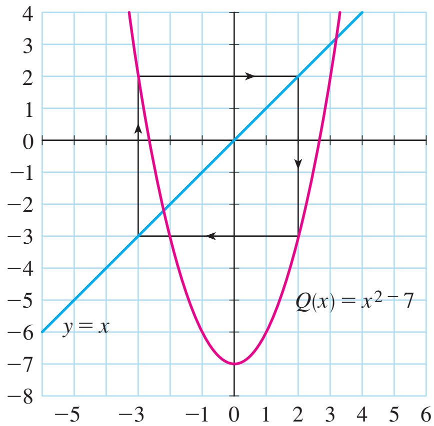


(c) In the following sentence, fill in the two blank spaces with numbers: The work in part (b) shows that {__, __} is a 2-cycle for the function T.

(d) In part (b) you used a graph to list the first six iterates of $x_0 = 0.4$ under the function $T(x) = 1 - |2x - 1|$. Now, using a calculator (or just simple arithmetic), actually compute x_1 and x_2 and thereby check your results.

34. As background for this exercise, you need to have worked part (a) in the previous exercise so that you know the definition of a 2-cycle.

(a) The curve in the following figure is the graph of the function $Q(x) = x^2 - 7$. Use the figure to list the first six iterates of -3 under the function Q. Also, list the first six iterates of 2.



(b) In the following sentence, fill in the two blank spaces with numbers: The work in part (a) shows that {__, __} is a 2-cycle for the function Q.

(c) In part (a) you used a graph to list the first six iterates of -3 under the function Q. Now, using the formula $Q(x) = x^2 - 7$, actually compute x_1 and x_2 and thereby check your results.

(d) Use your calculator to compute the first six iterates of $x_0 = -2.99$ under the function $Q(x) = x^2 - 7$. Note that the behavior of the iterates is vastly different than that observed in part (a), even though the initial inputs differ by only 0.01. (As pointed out in Exercise 28, this type of behavior is referred to as *sensitivity to initial conditions.*)

35. Let $f(x) = 4x(1 - x)$. In this exercise we find distinct inputs a and b such that $f(a) = b$ and $f(b) = a$. As indicated in Exercise 33, the set $\{a, b\}$ is called a *2-cycle for the function f.*

(a) From the equation $f(a) = b$ and the definition of f, we have

$$4a(1 - a) = b \tag{1}$$

Likewise, from the equation $f(b) = a$ and the definition of f, we have

$$4b(1 - b) = a \tag{2}$$

Subtract equation (2) from equation (1) and show that the resulting equation can be written

$$4(b - a)(b + a - 1) = b - a \tag{3}$$

(b) Divide both sides of equation (3) by the quantity $b - a$. (The quantity $b - a$ is nonzero because we are assuming that a and b are distinct.) Then solve the resulting equation for b in terms of a. You should obtain

$$b = \frac{5}{4} - a \tag{4}$$

(c) Use equation (4) to substitute for b in equation (1). After simplifying, you should obtain

$$16a^2 - 20a + 5 = 0 \tag{5}$$

(d) Use the quadratic formula to solve equation (5) for a. You should obtain

$$a = (5 \pm \sqrt{5})/8$$

(e) Using the positive root for the moment, suppose $a = (5 + \sqrt{5})/8$. Use this expression to substitute for a in equation (4). Show that the result is $b = (5 - \sqrt{5})/8$. Now check that these values of a and b satisfy the conditions of the problem. That is, given that $f(x) = 4x(1 - x)$, show that

$$f\left[\frac{1}{8}(5 + \sqrt{5})\right] = \frac{1}{8}(5 - \sqrt{5})$$

and

$$f\left[\frac{1}{8}(5 - \sqrt{5})\right] = \frac{1}{8}(5 + \sqrt{5})$$

Note: If we'd begun part (e) by using the other root of the quadratic, namely, $a = (5 - \sqrt{5})/8$, then we would have found $b = (5 + \sqrt{5})/8$, so no new information would have been obtained. In summary, the 2-cycle for the function f consists of the two numbers that are the roots of equation (5).

36. Let $f(x) = kx(1 - x)$ and assume that $k > 0$. Follow the method of Exercise 35 to show that the values of a and b for which $f(a) = b$ and $f(b) = a$ are given by the formulas

$$a = \frac{1 + k + \sqrt{(k - 3)(k + 1)}}{2k}$$

and

$$b = \frac{1 + k - \sqrt{(k - 3)(k + 1)}}{2k}$$

Note that for $k > 3$, both of these expressions represent real numbers (because the quantity beneath the radical sign is then positive). In summary: For all values of k greater than 3, the function $f(x) = kx(1 - x)$ has a unique 2-cycle $\{a, b\}$, where a and b are given by the preceding formulas.

37. **(a)** Let $f(x) = 3.5x(1 - x)$. Use the formulas in Exercise 36 to find values for a and b such that $\{a, b\}$ is a 2-cycle for this function [that is, so that $f(a) = b$ and $f(b) = a$]. Exact answers are required, not calculator approximations.

(b) The figure on the right shows the iteration process for the 2-cycle determined in part (a). Use the answers in part (a) to specify the coordinates of the four points P, Q, R, and S.

38. Let $f(x) = 3.2x(1 - x)$. Use the formulas in Exercise 36 to determine the values of a and b such that $\{a, b\}$ is a 2-cycle for this function. Use a calculator to evaluate the answers and round to three decimal places. [You'll find that these are the two numbers referred to in the caption for Figure 7(b) in this section.]

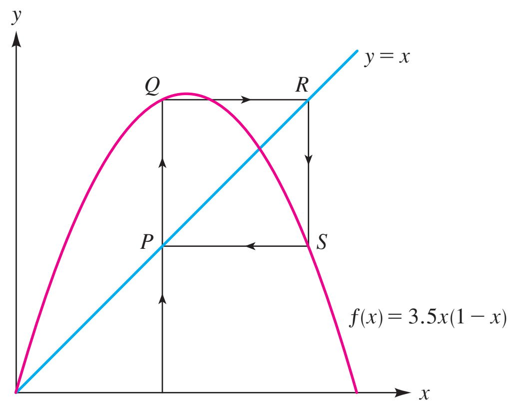


Figure for Exercise 37